Deterring Bribery with Scotch Hold ′em Poker

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Scandal!

- Dave the Developer requests permission for a new development from the Gav the Governor. He values planning permission at 1m Kč.
- Gav hires Ina the Inspector to examine Dave's proposed cite. He only grants permission if Ina doesn't find any evidence of endangered species.
- Ina is willing to coverup evidence in return for a bribe — creates surplus of 1.
- Marta the Mafia arbitrates bribery negotiations.

Gav can deter bribes by paying Ina $1+\epsilon$m to report evidence. Can Gav do better?

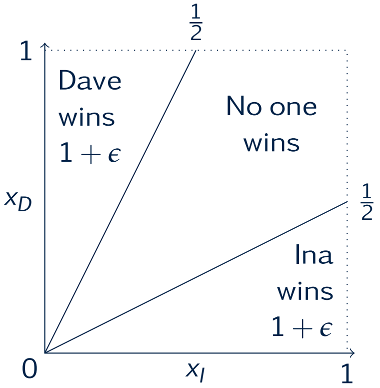
Randomness in Monitoring

- Randomness is useful for cutting down on monitoring costs:
  - in theory (Bentham's panopticon, Becker (1968); Ortner and Chassang (2018); von Negenborn and Pollrich (2020))
  - in practice (e.g. random timing of inspections, unknown inspector rank).
- We find a deeper role for randomising:
  - each player has private information to conceal their bargaining position.
  - the principal retains private information, so that even an arbitrator has trouble predicting (and undoing) the outcome.
  - incentives have a lemons market structure, to undermine the market for bribes.
- Our mechanism maybe implementable in practice, but we see it more as a starting point, e.g. for fostering competition between monitors.

Scotch Poker

- Gav deals
  - Dave a private hand $x_D \sim \mathcal{U}[0,1]$
  - Ina a private hand $x_I \sim \mathcal{U}[0,1]$.
- If Ina reports evidence then Gav denies permission and there is a showdown:
  - Gav pays Ina $1+\epsilon$ if $\frac{1}{2}x_I \geq x_D$ (Ina "wins") and 0 otherwise.
  - Gav pays Dave $1+\epsilon$ if $\frac{1}{2}x_D \geq x_I$ (Dave "wins") and 0 otherwise.
- Otherwise, Gav grants Dave permission.
  - Ina gets 0
  - Dave gets 1.

Scotch Poker



Scotch Hold 'em Poker

- Gav deals
  - Dave a private hand $x_D \sim \mathcal{U}[0,1]$
  - Ina a private hand $x_I \sim \mathcal{U}[0,1]$
  - a private community card $z \sim \mathcal{U}[0,1]$.
- If Ina reports evidence then Gav denies permission and there is a showdown:
  - Gav pays Ina $1+\epsilon$ if $zx_I \geq x_D$ (Ina "wins") and 0 otherwise.
  - Gav pays Dave $1+\epsilon$ if $zx_D \geq x_I$ (Dave "wins") and 0 otherwise.
- Otherwise, Gav grants permission.
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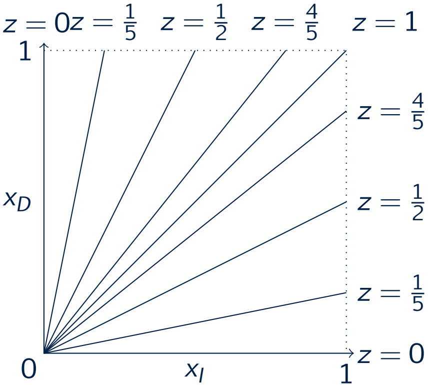
Scotch Hold ′em Poker



Results Preview

1. Scotch Poker deters simple bribes
   - it engineers an two-sided adverse selection problem in the market for coverups.
2. Scotch Poker doesn′t deter arbitrated bribes but Scotch Hold ′em Poker does
   - arbitration includes: Rubinstein bargaining, double auctions, side contracts ect.
   - the community card confounds the arbitrator.
3. Scotch Hold ′em Poker is "the best"
   - Gav pays out an average of $\frac{1}{2}$m Kč if Ina finds evidence.
   - he pays out at least as much in any other scheme that deters bribes.
4. "Crooked" Scotch Hold ′em Poker can deter bribes in moral hazard problems.
   - Gav can vary how much surplus each player gets.

Simple bribes

- Dave and Ina draw their hand.
- If Ina finds evidence then Marta proposes a bribe b.
  - If either Dave or Ina "reject" the bribe then
    - Ina reports the evidence
    - Gav initiates a showdown.
    - Ina gets $1 + \epsilon$ if she wins and Dave gets $1 + \epsilon$ if he wins; otherwise they get 0.
  - If Dave and Ina both "accept" the bribe then
    - Ina covers up evidence
    - Gav grants permission
    - Ina gets b and Dave gets $1 - b$.
- If Ina does not find evidence then there is no need for bribes.

Scotch Poker deters the equal share bribe.

Consider the equal share bribe $b = \frac{1}{2}$. In any equilibrium:

- Let $\bar{x}_D$ and $\bar{x}_I$ denote the supremum of hands for which Dave and Ina accept.
- They must accept for all hands below the supremum $\bar{x}_i$:
  - accepting forgoes the chance of winning and
  - the chance of winning is increasing in x_i.
- Ina's must prefer the bribe over the prize when Dave accepts she gets $\bar{x}_I$:

$$\frac{1}{2} \geq (1+\epsilon)\mathbb{P}[x_D \leq \tfrac{1}{2}\bar{x}_I | x_D \leq \bar{x}_D] = (1+\epsilon)\frac{\frac{1}{2}\bar{x}_I}{\bar{x}_D}$$

$$\implies \bar{x}_I \leq \bar{x}_D/(1+\epsilon) < \bar{x}_D$$

- Similar reasoning for Dave implies that $\bar{x}_D < \bar{x}_I$.
- But they can't both have lower threshold, unless one of them always rejects.

Scotch Poker deters all simple bribes.

In any equilibrium:

- Let $\bar{x}_D$ and $\bar{x}_I$ denote the supremum of hands for which Dave and Ina accept.
- They must accept for all hands below the supremum $\bar{x}_i$:
  - accepting forgoes the chance of winning and
  - their chance of winning is continuous and increasing in x_i.
- Ina's must prefer the bribe over the prize when she gets $\bar{x}_I$:

$$b \geq (1+\epsilon)\mathbb{P}[x_D \leq \tfrac{1}{2}\bar{x}_I | x_D \leq \bar{x}_D] = (1+\epsilon)\frac{\frac{1}{2}\bar{x}_I}{\bar{x}_D} > \frac{1}{2}\frac{\bar{x}_I}{\bar{x}_D}$$

- Similar reasoning for Dave implies that $1 - b > \frac{1}{2}\frac{\bar{x}_D}{\bar{x}_I}$.
- Together, they imply $b(1-b) > \frac{1}{4}$, a contradiction. So either $\bar{x}_D = 0$ or $\bar{x}_I = 0$.

Two-sided adverse selection

- Classic lemons (Akerlof, 1970): the seller has private information.
  - Unravelling occurs because of feedback between
    - buyer's expected value and
    - price.
  - Sellers with best goods leave the market $\Longrightarrow$ buyer's expected value falls $\Longrightarrow$ price falls $\Longrightarrow$ ...
- Two-sided lemons: the buyer and the seller both have private information.
  - Unravelling occurs at each individual price level because of feedback between
    - the buyer's expected value of the good, and
    - the seller's expected value of the good.
  - seller types with the highest expected value of the good leave the market $\Longrightarrow$ buyer's expected value of the good falls $\Longrightarrow$ buyer types with the lowest expected value leave the market $\Longrightarrow$ seller's expected value of the good increases $\Longrightarrow$ next highest sellers leave $\Longrightarrow$ ...
- The logic is similar to the "winner's curse" in auction theory.

Arbitrated Bribes

- Scotch poker halves the cost of deterring bribes by engineering a two-sided adverse selection problem.
- But Marta can easily solve this market failure and hence undermine the mechanism (next slide).
- Scotch Hold 'Em Poker fixes this vulnerability by adding a "community card" that confounds the mafia, and every other negotiation protocol (Myerson, 1981).

Scotch Poker does not deter arbitrated bribes.

- Suppose Marta offers the following side contract:
  - Dave and Ina privately show their hands to Marta.
  - If both having losing hands, then Marta tells Ina to coverup the evidence and charges Dave a fee of 1.
  - Otherwise, if a players has a winning hand, she tells Ina to report the evidence.
- The contract is incentive compatible:
  - If both have losing hands, then
    - Dave gets ϵ if he reports any losing hand, and 0 otherwise.
    - Ina gets 0, no matter what she reports.
  - If either player has a winning hand, then they get $1 + \epsilon$ if they report any winning hand, and 0 otherwise.
- The contract is strictly profitable for Marta: she gets
  - 1 if both hands are losing (probability $\frac{1}{2}$)
  - 0 otherwise.

Scotch Hold 'em Poker deters arbitrated bribes (overview)

- It is without loss of generality for Marta to restrict attention to direct, truthful and voluntary contracts (Myerson, 1981).
  - Truthful (incentive compatible): players prefer to report their hand truthfully
  - Voluntary (individually rational): players prefer to accept the contract.
- A side contract (a, b_I, b_D) specifies for each pair of hands $x = (x_I, x_D)$
  - a probability $a(x)$ with which to coverup evidence (a for "allocation")
  - payments $b_I(x)$ to Ina and $b_D(x)$ to Dave (b for "bribe").
- We show that the cost of incentivising truth telling exceeds the surplus of coordinating.

Scotch Hold 'em Poker deters arbitrated bribes (details 1/3)

Suppose $\epsilon = 0$ for notational simplicity.

- Player i's expected utility from reporting evidence, conditional on x:

$$\mathbb{P}[zx_i \geq x_j] = \mathbb{P}[z \geq \tfrac{x_j}{x_i}] = 1 - \min\left\{1, \tfrac{x_j}{x_i}\right\} = \max\left\{0, 1 - \tfrac{x_j}{x_i}\right\}.$$

- Player i's expected utility when their true hand is x_i and they report hand x_i':

$$V_i(x_i'; x_i) := \int_0^1 a(x_i', x_j)\mathcal{I}(i = D) + (1 - a(x_i', x_j)) \max\left\{0, 1 - \tfrac{x_j}{x_i}\right\} + b_i(x_i', x_j)\, dx_j$$

- Player i's utility (information rent) when they get hand x_i:

$$W_i(x_i) := \max_{x_i' \in [0,1]} V_i(x_i'; x_i)$$

$$\frac{d}{dx_i} W_i(x_i) \overset{\text{ET}}{=} \frac{\partial}{\partial x_i} V_A(x_i'; x_i)\bigg|_{x_i' \overset{\text{IC}}{=} x_i} = -\frac{1}{x_i} \int_0^{x_i} a(x_i, x_j) \tfrac{x_j}{x_i}\, dx_j$$

Scotch Hold 'em Poker deters arbitrated bribes (details 2/3)

Dave's expected utility from the contract is

$$\int_0^1 W_D(x_D)\,dx_D \overset{\text{FTC}}{=} \int_0^1 W_D(1) - \int_{x_D}^1 \frac{d}{dy_D} W_D(y_D)\,dy_D\,dx_D$$

$$\overset{\text{VP1}}{\geq} -\int_0^1 \int_{x_D}^1 \frac{d}{dy_D} W_D(y_D)\,dy_D\,dx_D$$

$$\overset{\text{ET}}{=} \int_0^1 \underbrace{\int_{x_D}^1 \int_0^{y_D} a(y_D, x_I) \frac{x_I}{y_D^2}\,dx_I\,dy_D}_{1\times g(x_D)}\,dx_D$$

$$\overset{\text{IBP}}{=} \int_0^1 \underbrace{\int_0^{x_D} a(x_D, x_I) \frac{x_I}{x_D}\,dx_I}_{-x_D\times g'(x_D)}\,dx_D$$

Scotch Holdem Poker deters arbitrated bribes (details 3/3)

- Similarly, Ina's expected utility is

$$\int_0^1 W_I(x_I)\,dx_I \geq \int_0^1 \int_0^{x_I} a(x_D, x_I)\frac{x_D}{x_I}\,dx_D\,dx_I = \int_0^1 \int_{x_D}^1 a(x_D, x_I)\frac{x_D}{x_I}\,dx_I\,dx_D.$$

- Their joint expected utility is

$$\int_0^1 W_D(x_D)\,dx_D + \int_0^1 W_I(x_I)\,dx_I \geq \int_0^1 \int_0^1 a(x_D, x_I)\min\left\{\frac{x_D}{x_I}, \frac{x_I}{x_D}\right\}\,dx_D\,dx_I.$$

- The total expected net surplus from coverups is

$$1 - \mathbb{P}\left(z \geq \frac{x_I}{x_D}\right) - \mathbb{P}\left(z \geq \frac{x_D}{x_I}\right) = \min\left\{\frac{x_I}{x_D}, \frac{x_D}{x_I}\right\}$$

so the mafia cannot make a profit. $\epsilon > 0$ breaks indifferences. □

Optimality

- We have shown that our mechanism deters bribery in a very robust sense.
- Finally, we show that it is *the cheapest way* to deter bribes.
- In doing so, we provides three further methodological insights:

1. Carroll (2016) shows that private information does not harm trade at a fixed price, but we show that it harms player's ability to coordinate on a price.
2. Our mechanism engineers an extreme payoff-information structure that delimits the most unfavourable conditions for coordination.
3. We generalise proof techniques for information structures with a finite number of types by circumventing a transfinite induction problem.

Schemes

- Scotch Hold 'em Poker is a special kind of *scheme*.
- Generally, a scheme $\mathcal{S} = (X, Y, \Sigma, p, t)$ consists of
  - message spaces (possible hands) X and Y, one for Ina, one for Dave.
  - a sigma-algebra Σ on $X \times Y$.
  - a probability measure p
  - a pair of p-measurable transfer functions $t := (t_I, t_D)$.
- The cost of a scheme $\mathcal{S}$ is the total expected transfer $\mathbb{E}_p[t_I(x,y) + t_D(x,y)]$.

Scotch Hold 'em Poker is Best: proof overview

Any scheme that deters bribery costs at least as much as Scotch Hold 'em Poker.

1. Define what it means for a scheme to "infect a hand" and to start an "outbreak".
2. Show that any scheme that deters the equal share bribe necessarily infects all possible hands of at least one player.
3. Show that any scheme that infects all the possible hands of either player necessarily costs at least $\frac{1}{2}$, i.e. the same as Scotch Hold 'em Poker.

Infection

- Suppose Marta proposes the (simple) equal share bribe $b = \frac{1}{2}$.
- Dave and Ina each choose a measurable acceptance strategy $a_D : X \to [0, 1]$ and $a_I : Y \to [0, 1]$ respectively.
  - if they both accept then Dave pays Ina $\frac{1}{2}$ and Ina covers up the evidence
    - they each get $\frac{1}{2}$.
  - otherwise, Ina reports the evidence and they receive transfers $t_I(x)$ and $t_D(y)$.
- If player i's is better off rejecting when they have hand x_i and player j plays a_j, then we say that "strategy a_j infects hand x_i".

Outbreaks I

Let $\mathcal{S}$ be any scheme. A *finite outbreak* of length N is a pair of finite sequences of infected hands $(X_n, Y_n)_{n=0}^{N} \subseteq (X \times Y)^N$ that satisfy the following 4 properties:

1. At the start, no hands are infected:
   - $X_0 = Y_0 = \emptyset$.
2. Once infected, hands stay infected:
   - $X_n \subseteq X_{n+1}$ and $Y_n \subseteq Y_{n+1}$.
3. In round n, at most one player has a non-empty set of *newly infected hands*:
   - either $\bar{X}_n := X_n \backslash X_{n-1} \neq \emptyset$ or $\bar{Y}_n := X_n \backslash X_{n-1} \neq \emptyset$, but not both.

Outbreaks II

4. Newly infected player i hands get infected by an "adverse player j strategy":
   - if $\bar{X}_n \neq \emptyset$ then there exists a strategy a_I such that
     1. uninfected hands accept: $a_I(y) = 1$ for all $y \notin Y_n$;
     2. Dave's newly infected hands are better off rejecting:

$$\int_Y \left[t_D(x,y) - \tfrac{1}{2}\right] a_I(y)\, dp(y|x) > 0 \qquad \forall x \in \bar{X}_n.$$

   - similarly, if $\bar{Y}_n \neq \emptyset$ then there exists a strategy a_D such that $a_D(x) = 1$ for all $x \notin X_n$, and

$$\int_X [t_I(x,y) - \tfrac{1}{2}] a_D(x)\, dp(x|y) > 0 \qquad \forall y \in \bar{Y}_n.$$

An *infinite outbreak* $(X_n, Y_n)_{n \in \mathbb{N}}$ is defined similarly.

Deterring bribes implies Complete Outbreak

Lemma

If $\mathcal{S}$ deters the equal share bribe then there exists an infinite outbreak $(X_n, Y_n)_{n \in \mathbb{N}}$ that infects all of Ina's hands, i.e. $p(X_n \times Y) \to 1$.

Proof.

- Finite message spaces: a corollary of Carroll (2016, Propositions 3.1 and 3.2).
- Infinite message spaces: we sidestep a transfinite induction problem (our methodological contribution).

□

Deterring bribes implies Complete Outbreak: proof (finite)

1. Suppose an outbreak leaves some of both players' hands uninfected.
2. The "constrained" game in which uninfected hands are constrained to accept and infected type are free to mix between accepting and rejecting. has an equilibrium (Nash existence theorem).
3. Uninfected hands exchange bribes in this equilibrium, so it cannot be an equilibirum of the unconstrained game.
4. Hence, some of the uninfected types must best respond the the constrained equilibrium strategies by rejecting bribes $\implies$ the outbreak can be extended.
5. If type spaces are finite then induction implies, WLOG, that all of Ina's hands get infected.

Deterring bribes implies Complete Outbreak: proof (infinite)

The "size" of an outbreak is the measure of Ina's hands that it infects.

- **Claim 1:** there exists an infinite outbreak that is the same size as the supremum of the size of all the finite outbreaks.
- **Claim 2:** any infinite outbreak that fails to infect all of the buyer hands is smaller than some finite outbreak.
- Hence the supremum of the size of all the finite outbreaks is 1, otherwise
  - claim 1 gives an infinite outbreak that attains the supremum,
  - claim 2 gives a larger finite outbreak that exceeds the supremum — a contradiction.
- Claim 1 implies that there exists an infinite outbreak of size 1,
  - $\implies$ it infects almost all of Ina's hands.

Complete Outbreak implies a cost of $\frac{1}{2}$

Lemma

The expected cost of infecting all of Ina's hands is at least $\frac{1}{2}$.

Ina's round n infected hands prefer their expected transfers over the bribe $\frac{1}{2}$:

$$\begin{aligned}
\int_{\bar{X}_{n+1}\times Y} t_I(x,y)\,dp(x,y) &\geq \int_{\bar{X}_{n+1}\times Y} a_D^n(y)t_I(x,y)\,dp(x,y) \\
&\geq \int_{\bar{X}_{n+1}\times Y} a_D^n(y)\frac{1}{2}\,dp(x,y) \\
&\geq \frac{1}{2}\int_{\bar{X}_{n+1}\times Y} \mathcal{I}(y \in Y \setminus Y_n)\,dp(x,y) \\
&\geq \frac{1}{2}p(\bar{X}_{n+1} \times Y \setminus Y_n)
\end{aligned}$$

Complete Outbreak implies a cost of $\frac{1}{2}$: proof 2/3

Similarly for Dave,

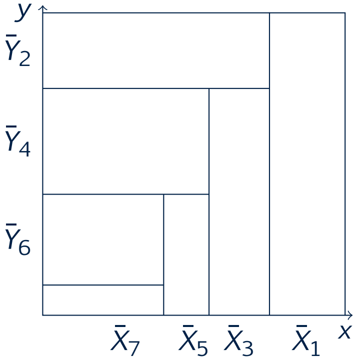
$$\int_{X\times\bar{Y}_{n+1}} t_D \, dp \geq \frac{1}{2} p((X \backslash X_n) \times \bar{Y}_{n+1}).$$

Total expected transfers are thus

$$\begin{aligned}\int_{X\times Y}(t_B + t_D)\, dp &\geq \sum_{n=1}^{\infty}\int_{\bar{X}_{n+1}\times Y} t_B \, dp + \sum_{n=1}^{\infty}\int_{X\times\bar{Y}_{n+1}} t_D \, dp \\ &\geq \frac{1}{2}\sum_{n=1}^{\infty}\Big(p(\bar{X}_{n+1}\times(Y\backslash Y_n)) + p((X\backslash X_n)\times\bar{Y}_{n+1})\Big).\end{aligned}$$

Complete Outbreak implies a cost of $\frac{1}{2}$: proof 3/3

The sets $\{\bar{X}_{n+1} \times (Y \setminus Y_n)\}_{n \in \mathbb{N}}$ and $\{(X \setminus X_n) \times \bar{Y}_{n+1}\}_{n \in \mathbb{N}}$ cover the whole space.



So $\int_{X \times Y} (t_B + t_D)\, dp \geq \frac{1}{2}$ as required.

Conclusion

1. We **design a general mechanism** to **minimise the cost** of **deterring bribes.**
   - Corruption: Ortner and Chassang (2018); Baliga and Sjöström (1998); Laffont and Martimort (1997); Tirole (1986); Strausz (1997).
   - Information design: Condorelli and Szentes (2020); Garrett et al. (2021).
   - General mechanism design: Halac et al. (2021); Schottmüller (2021).
2. We break the market for corruption by **engineering a lemons problem**.
   - The market for lemons: Akerlof (1970)
   - Endogenous lemons: von Negenborn and Pollrich (2020)
3. We study a **worst case, two-sided lemons problem.**
   - Worst case information: Carroll (2016)
   - Contageous adverse selection: Morris and Shin (2012).

Extensions

- Unequal split of surplus (Crooked Scotch Hold ′em Poker)
  - incentivising the agent with imperfect monitoring
  - monitoring costs
- Multiple players (next)
  - monitors/whistleblowers
  - bidders
- No hard evidence/costly evidence fabrication (future).

Crooked Scotch Hold ’em Poker

- Gav deals
  - Dave a private hand $x_D \sim \mathcal{U}[0,1]$
  - Ina a private hand $x_I \sim \mathcal{U}[0,1]$
  - a private community card $z \sim \mathcal{U}[0,1]$.
- If Ina reports evidence then Gav denies permission and there is a showdown:
  - Gav pays Ina $1+\epsilon$ if $zx_I \geq x_D^{\lambda}$ (Ina “wins”) and 0 otherwise
  - Gav pays Dave $1+\epsilon$ if $zx_D \geq x_I^{1/\lambda}$ (Dave “wins”) and 0 otherwise.
  - Expected surplus: Ina gets $\frac{\lambda^2}{(1+\lambda)^2}$ and Dave gets $\frac{1}{(1+\lambda)^2}$
- Otherwise, Gav grants permission.
  - Ina gets 0
  - Dave gets 1.

Thank you!

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Claim

Let $\mathcal{S}$ be a scheme and let O denote the set of finite outbreaks. Let $r^* = \sup_{(X_n, Y_n)_{n=0}^N \in O} p_B(X_N)$ denote the supremum of the size of all the finite outbreaks. There exists an infinite outbreak of size r^*.

Proof.

Since r^* is the supremum, there exists a sequence of finite outbreaks $(X^m, Y^m)_{m \in \mathbb{N}}$, each of length N^m, such that $r^m = p_B(X^m_{N^m})$ converges to r^*. Let X'_n be the concatenation of these sequences, i.e. $X' = X^0 \,\|\, X^1 \,\|\, \cdots$. Let X''_n be the sequence $X''_0 = X'_0$ and $X''_{n+1} = X'_{n+1} \cup X''_n$. Construct the sequences Y'_n and Y''_n in the same way. The sequence $(X''_n, Y''_n)_{n \in \mathbb{N}}$ is an infinite outbreak. Every infection set at the end of each outbreak, $X^m_{N^m}$, is contained as a subset of some set X''_n, it follows that $p_B(X''_n) \to r^*$. □

Claim

Let $(X_n, Y_n)_{n\in\mathbb{N}}$ be an infinite outbreak of size $r := \lim_{n\to\infty} p_B(X_n)$ that falls short of infecting all of the buyer hands, so that $r < 1$. There exists a finite outbreak $(X'_n, Y'_n)_{n\leq N}$ with size $p_B(X'_N) > r$.

Let $(X^*, Y^*) = (\cup_{n\in\mathbb{N}} X_n, \cup_{n\in\mathbb{N}} Y_n)$ denote the set of hands infected by the outbreak, so that $p_B(X^*) = r$. Consider the amended game in which the uninfected hands $(X\backslash X^*, Y\backslash Y^*)$ are constrained to accept trade, but the remaining hands may choose to either accept or reject the trade. [1] The assumptions above ensure that of Balder's equilibrium existence theorem (Balder, 1988, Theorem 3.1) apply, so an equilibrium (a_B^*, a_D^*) exists in the constrained game. There are no profitable deviations within this restricted strategy space. But trade between uninfected hands occurs, so (a_B^*, a_D^*) is not an equilibrium in the unconstrained game. Therefore, the buyer (without loss of generality) must have a profitable deviation by a strictly positive measure of uninfected hands $\bar{X} \subseteq X \setminus X^*$. These hands get infected in the unconstrained game, so they would rather reject trade, i.e. for all $x \in \bar{X}$,

$$\int_Y [T_B(x) - (1-b)] a_D^*(y) dp(y|x) > 0. \tag{1}$$

Summing up, we deduce

$$\int_{\bar{X} \times Y} [T_B(x) - (1-b)] a_D^*(y) dp(x,y) > 0. \tag{2}$$

Since uninfected seller hands $y \in Y \setminus Y^*$ accept, we can write

$$\int_{\bar{X} \times Y^*} a_D^*(y) [T_B(x) - (1-b)] dp(x,y) + \int_{\bar{X} \times (Y \setminus Y^*)} [T_B(x) - (1-b)] dp(x,y) > 0. \tag{3}$$

Now, for each period $N \in \mathbb{N}$ in which buyer hands get infected (not seller hands), consider the finite sequence $(X'_n, Y'_n)_{n \leq N}$ which equals (X_n, Y_n) up until period $N-1$, and equals $(X_{N-1} \cup \bar{X}, Y_N)$ in period N. In period N, the seller plays a strategy a_D^N which is equal to 1 (trade) on the set $Y \setminus Y_N \supseteq Y \setminus Y^*$ (so it satisfies the constrainr in round N), and which is equal to a_D^* on the set Y_N. The payoff for buyer hands in $\bar{X}$ under a_D^N differs from their payoff under a_D^* by

$$\int_{Y^* \setminus Y_N} |1 - a_D^*(y)| [T_B(x) - (1-b)] \, dp(y|x), \tag{4}$$

which is at most $\mathbb{P}[Y^* \setminus Y_N] \times [T_B(x) - (1-b)] \xrightarrow{N \to \infty} 0$. hands in $\bar{X}$ get a strictly prefer to reject trade under a_D^*, so there must exists some N large enough that they strictly prefer to reject trade under a_D^N as well. This means that hands in $\bar{X}$ can get infected in round N, so $(X'_n, Y'_n)_{n \leq N}$ is a well defined finite outbreak.

Moreover, X_{N-1} and $\bar{X}$ are disjoint, so the size of $(X'_n, Y'_n)_{n\leq N}$ is

$$p_B(X_{N-1} \cup \bar{X}) = p_B(X_{N-1}) + p_B(\bar{X}) \to r + p(\bar{X}),$$

which is strictly greater than r. Thus, there exists a large enough N that defines a finite outbreak $(X'_n, Y'_n)_{n\leq N}$ with size greater than r.

[1] We owe this proof technique to Carroll (2016)'s study of the finite case.